

ON THE CLASSICAL MAIN CONJECTURE FOR IMAGINARY QUADRATIC FIELDS

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January 13, 2013

Abstract

Let p be a prime number, and let k be an imaginary quadratic number field in which p decomposes into two distinct primes \mathfrak{p} and $\bar{\mathfrak{p}}$. Let k_∞ be the unique \mathbb{Z}_p -extension of k which is unramified outside of \mathfrak{p} , and let K_∞ be a finite extension of k_∞ , abelian over k . In case $p \notin \{2, 3\}$, we prove that in K_∞ , the characteristic ideal of the projective limit of the p -class group coincides with the characteristic ideal of the projective limit of units modulo elliptic units. Our approach is based on Euler systems, which were first used in this context by Rubin in [14]. For $p \in \{2, 3\}$, we obtain a divisibility relation, up to a certain constant.

Mathematics Subject Classification (2010): 11G16, 11R23, 11R65.

Key words: Elliptic units, Euler systems, Iwasawa theory.

1 Introduction.

Let p be a prime number, and let k be an imaginary quadratic number field in which p decomposes into two distinct primes \mathfrak{p} and $\bar{\mathfrak{p}}$. Let k_∞ be the unique \mathbb{Z}_p -extension of k which is unramified outside of \mathfrak{p} , and let K_∞ be a finite extension of k_∞ , abelian over k . Let G_∞ be the Galois group of K_∞/k . We choose a decomposition of G_∞ as a direct product of a finite group G (the torsion subgroup of G_∞) and a topological group Γ isomorphic to \mathbb{Z}_p , $G_\infty = G \times \Gamma$. For any $n \in \mathbb{N}$, let K_n be the field fixed by $\Gamma_n := \Gamma^{p^n}$, and let $G_n := \text{Gal}(K_n/k)$. Remark that there may be different choices for Γ , but when p^n is larger than the order of the p -part of G , the group Γ_n does not depend on the choice of Γ .

Let F/k be an abelian extension of k . If $[F : k] < \infty$, we denote by \mathcal{O}_F the ring of integers of F . We write \mathcal{O}_F^\times for the group of global units of F , and C_F for the group of elliptic units of F (see section 3). Also we let A_F be the p -part of the class group $\text{Cl}(\mathcal{O}_F)$ of \mathcal{O}_F . We set $\mathcal{E}_F := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F^\times$ and $\mathcal{C}_F := \mathbb{Z}_p \otimes_{\mathbb{Z}} C_F$. When F/k is infinite, we define \mathcal{E}_F , \mathcal{C}_F and A_F , by taking projective limits over finite sub-extensions, under the norm maps. For any $n \in \mathbb{N} \cup \{\infty\}$, we set $\mathcal{E}_n := \mathcal{E}_{K_n}$, $\mathcal{C}_n := \mathcal{C}_{K_n}$, and $A_n := A_{K_n}$.

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For any profinite group \mathcal{G} , and any commutative ring R , we define the Iwasawa algebra

$$R[[\mathcal{G}]] := \varprojlim R[\mathcal{H}],$$

where the projective limit is over all finite quotient \mathcal{H} of \mathcal{G} . In case $\mathcal{G} = G_\infty$, we shall write

$$\Lambda := \mathbb{Z}_p[[G_\infty]].$$

Then A_∞ and $\mathcal{E}_\infty/\mathcal{C}_\infty$ are naturally Λ -modules. As we shall see below, they are finitely generated and torsion over $\mathbb{Z}_p[[\Gamma]]$. Let us fix a topological generator γ of Γ , and set $T := \gamma - 1$. Then for any finite extension L/\mathbb{Q}_p , $\mathcal{O}_L[[\Gamma]]$ is isomorphic to $\mathcal{O}_L[[T]]$, where \mathcal{O}_L is the ring of integers of L . It is well known that $\mathcal{O}_L[[T]]$ is a noetherian, regular, local domain. We also recall that $\mathcal{O}_L[[T]]$ is a unique factorization domain, and that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_L[[T]]$ is a principal ring. If u is a uniformizer of \mathcal{O}_L , then the maximal ideal \mathfrak{m} of \mathcal{O}_L is generated by u and T , and $\mathcal{O}_L[[T]]$ is a compact topological ring with respect to its \mathfrak{m} -adic topology. A morphism $f : M \rightarrow N$ between two finitely generated $\mathcal{O}_L[[T]]$ -module is called a pseudo-isomorphism if its kernel and its cokernel are finite. If a finitely generated $\mathcal{O}_L[[T]]$ -module M is given, then one may find elements P_1, \dots, P_r in $\mathcal{O}_L[T]$, irreducible in $\mathcal{O}_L[[T]]$, and nonnegative integers n_0, \dots, n_r , such that there is a pseudo-isomorphism

$$M \longrightarrow \mathcal{O}_L[[T]]^{n_0} \oplus \bigoplus_{i=1}^r \mathcal{O}_L[[T]] / (P_i^{n_i}).$$

Moreover, the integer n_0 and the ideals $(P_1^{n_1}), \dots, (P_r^{n_r})$, are uniquely determined by M . If $n_0 = 0$, then the ideal generated by $P_1^{n_1} \cdots P_r^{n_r}$ is called the characteristic ideal of M , and is denoted by $\text{char}_{\mathcal{O}_L[[T]]}(M)$.

We denote by \mathbb{C}_p a completion of an algebraic closure of \mathbb{Q}_p . Let $\chi : G \rightarrow \mathbb{C}_p^\times$ be an irreducible character of G . Let $\mathbb{Q}_p(\chi) \subset \mathbb{C}_p$ be the abelian extension of \mathbb{Q}_p generated by the values of χ . We denote by $\mathbb{Z}_p(\chi)$ the ring of integers of $\mathbb{Q}_p(\chi)$. The group G acts naturally on $\mathbb{Q}_p(\chi)$. We recall that if $g \in G$ and $x \in \mathbb{Q}_p(\chi)$ then $g.x := \chi(g)x$. For any $\mathbb{Z}_p[G]$ -module Y , we define its χ -quotient $Y_\chi := \mathbb{Z}_p(\chi) \otimes_{\mathbb{Z}_p[G]} Y$. Moreover, if we set

$$\psi : G \longrightarrow \mathbb{Q}_p, \quad \sigma \mapsto \text{Tr}(\chi(\sigma)),$$

where Tr is the trace map for the extension $\mathbb{Q}_p(\chi)/\mathbb{Q}_p$, then ψ is an irreducible \mathbb{Q}_p character on G , and we will write $\chi|\psi$. Recall that any irreducible \mathbb{Q}_p character on G can be obtained in this way. We define the idempotent e_ψ of $\mathbb{Q}_p[G]$ attached to ψ ,

$$e_\psi = \frac{1}{\#G} \sum_{g \in G} \psi(g) g^{-1}.$$

The restriction to the ψ -part of the canonical surjective map $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y \rightarrow \mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y_\chi$ is an isomorphism of $\mathbb{Q}_p[G]$ -modules,

$$e_\psi (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y) \simeq \mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y_\chi \simeq \mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[G]} Y. \quad (1.1)$$

Also, we have the following decomposition

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y = \bigoplus_{\psi} e_\psi (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y),$$

where the sum is over all irreducible \mathbb{Q}_p characters ψ on G . Finally, if Y is a Λ -module, then Y_χ is a $\mathbb{Z}_p(\chi)[[T]]$ -module in a natural way. As a particular case, $\Lambda_\chi \simeq \mathbb{Z}_p(\chi)[[T]]$. For any finitely generated Λ_χ -module Z , we shall denote $\text{char}_{\Lambda_\chi} Z$ simply by $\text{char} Z$.

The goal of this article is to prove Theorem 1.1 below, which is a formulation of the (one-variable) main conjecture. In [15, Theorem 4.1] and [16, Theorem 2], Rubin used Euler systems to prove the main conjectures for \mathbb{Z}_p or \mathbb{Z}_p^2 extensions of a finite abelian extension F of k , where $p \nmid w_k[F : k]$, w_k being the number of roots of unity in k . More recently, Hassan Oukhaba adapted Rubin's method and obtained Theorem 1.1 for $p = 2$, still under the condition $2 \nmid [K_0 : k]$ (see [10]). Inspired by the ideas of Rubin, Greither used Euler systems to prove the main conjecture for cyclotomic units and for the cyclotomic \mathbb{Z}_p -extension F_∞/F , with F_∞ abelian over \mathbb{Q} (see [4, Theorem 3.2]). Bley proved Theorem 1.1 when $p \nmid 2\#(\text{Cl}(\mathcal{O}_k))$, and when there is a nonzero ideal \mathfrak{f} of \mathcal{O}_k , prime to \mathfrak{p} , such that for all $n \in \mathbb{N}$, $K_n = k(\mathfrak{f}\mathfrak{p}^{n+1})$ is the ray class field of k modulo $\mathfrak{f}\mathfrak{p}^{n+1}$ (see [1, Theorem 3.1]). Here we prove the general case.

Also, we draw the attention of the reader to a cohomological two-variables main conjecture, which has been recently proved for all primes by J. Johnson-Leung and G. Kings in [7], as a consequence of the Tamagawa number conjecture. In their treatment they replaced χ -quotients by Galois cohomology with coefficients in the Galois representations defined by χ , and used Euler systems as defined by Kato. From their result, they deduced the classical two-variables main conjecture for \mathbb{Z}_p^2 -extensions $F_\infty := \bigcup_{n=0}^{\infty} k(p^n\mathfrak{f})$ where \mathfrak{f} is any nonzero ideal of \mathcal{O}_k , and when p does not divide the torsion subgroup of $\text{Gal}(F_\infty/k)$.

Theorem 1.1 *Let χ be an irreducible \mathbb{C}_p character on G .*

- (i) *If $p \notin \{2, 3\}$, then $\text{char}(A_{\infty, \chi}) = \text{char}(\mathcal{E}_\infty/\mathcal{C}_\infty)_\chi$.*
- (ii) *If $p \in \{2, 3\}$, then there is $m_\chi \in \mathbb{N}$ such that*

$$\text{char}(A_{\infty, \chi}) \quad \text{divides} \quad \mathfrak{u}_\chi^{m_\chi} \text{char}(\mathcal{E}_\infty/\mathcal{C}_\infty)_\chi, \quad (1.2)$$

where \mathfrak{u}_χ is a uniformizer of $\mathbb{Z}_p(\chi)$.

2 Semi-local units.

For every $n \in \mathbb{N}$, we denote by \mathcal{U}_n the $\mathbb{Z}_p[G_n]$ -module of principal semi-local units over the primes above \mathfrak{p} . We define

$$\mathcal{U}_\infty := \varprojlim \mathcal{U}_n,$$

by taking the projective limit under the norm maps.

For any $n \in \mathbb{N}$, we write γ_n for γ^{p^n} . Then for any $\mathbb{Z}_p[[T]]$ -module M , we denote by M^{Γ_n} the module of Γ_n -invariants of M , and we denote by M_{Γ_n} the module of Γ_n -coinvariants of M . By definition, they are respectively the kernel and the cokernel of the multiplication by $1 - \gamma_n$ on M .

Proposition 2.1 *$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}_\infty$ is a free $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda$ -module of dimension 1.*

Proof. Let \mathfrak{P} be a prime of K_∞ above \mathfrak{p} , and let k' be the completion of k at \mathfrak{p} . For every $n \in \mathbb{N}$ we respectively denote by K'_n , \mathcal{O}'_n , and $\widehat{\mathfrak{P}}_n$ the completion of K_n at \mathfrak{P} , the ring of integers of K'_n , and the maximal ideal of \mathcal{O}'_n . Let us also denote the group $1 + \widehat{\mathfrak{P}}_n$ by \mathcal{U}'_n . For sufficiently large m , the p -adic logarithm is an isomorphism of $\mathbb{Z}_p[\text{Gal}(K'_n/k')]$ -modules from $1 + \widehat{\mathfrak{P}}_n^m$ into $\widehat{\mathfrak{P}}_n^m$. Taking the tensor product of this isomorphism by \mathbb{Q}_p over \mathbb{Z}_p , we see that

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}'_n \simeq K'_n \simeq \mathbb{Q}_p[\text{Gal}(K'_n/k')], \quad (2.1)$$

where the last isomorphism holds by the normal basis theorem, and since $k' = \mathbb{Q}_p$. The field $K'_\infty := \bigcup_{n=0}^{\infty} K'_n$ is a \mathbb{Z}_p -extension of K'_0 , abelian over k' . The Galois group of K'_∞/k' is canonically identified to the decomposition group of \mathfrak{p} in K_∞/k . Hence we only have to show that the projective limit

$$\mathcal{U}'_\infty := \varprojlim \mathcal{U}'_n,$$

with respect to the norm maps, is such that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}'_\infty$ is a free $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda'$ -module, where

$$\Lambda' := \mathbb{Z}_p[[\text{Gal}(K'_\infty/k')]].$$

We have a decomposition $\text{Gal}(K'_\infty/k') = G' \times \Gamma'$, where $G' \subseteq G$, and $\Gamma' \simeq \mathbb{Z}_p$ as a pro- p -group. Remark that for n large enough, we have $\Gamma_n \subseteq \Gamma'$. Let χ be any \mathbb{C}_p irreducible character on G' , and let $\psi : G' \rightarrow \mathbb{Q}_p$ be the unique irreducible character such that $\chi|\psi$. We must prove that $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}'_\infty)$ is a free $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda')$ -module. Since the group of p -power roots of unity in K'_∞ is finite by Lemma 2.1 below, we deduce from [5, Theorem 25] that $\mathcal{U}'_\infty \hookrightarrow \mathbb{Z}_p[[\Gamma']]^d$ where $d = [K'_0 : k']$. In particular $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}'_\infty) \subseteq (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma']])^d$ and hence is a torsion free $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda')$ -module. Let us remark that $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda') \simeq \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi)[[\Gamma']]$ is a principal ring. We deduce that $e_\psi \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}'_\infty$ is a free $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda')$ -module. Let r_χ be its rank. We must show that $r_\chi = 1$. Let us choose n such that $\Gamma_n \subseteq \Gamma'$. Then

$$e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\mathcal{U}'_\infty)_{\Gamma_n}) \simeq e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\text{Gal}(K'_n/k)])^{r_\chi}. \quad (2.2)$$

Exactly as in [10, Proof of Proposition 3.6], one can prove that the kernel and the cokernel of the canonical map $(\mathcal{U}'_\infty)_{\Gamma_n} \rightarrow \mathcal{U}'_n$ are finitely generated \mathbb{Z}_p -modules of rank 1, and invariant under the action of $\text{Gal}(K'_\infty/k')$. We deduce that

$$\dim_{\mathbb{Q}_p}(e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\mathcal{U}'_\infty)_{\Gamma_n})) = \dim_{\mathbb{Q}_p}(e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}'_n)). \quad (2.3)$$

But $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}'_n) \simeq e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\text{Gal}(K'_n/k)])$ by (2.1). Thus $r_\chi = 1$ by (2.3) and (2.2). \square

Lemma 2.1 *Let the notation be as in the proof of Proposition 2.1. Then the group $\mu_{p^\infty}(K'_\infty)$ of p -power roots of unity in K'_∞ is finite.*

Proof. As previously, we write k' for the completion of k at \mathfrak{p} . Since $k' = \mathbb{Q}_p$, it is well known that the kernel of the local norm residue symbol

$$(\cdot, k'(\mu_{p^\infty})/k') : (k')^\times \rightarrow \text{Gal}(k'(\mu_{p^\infty})/k')$$

is the free group $\langle p \rangle$ generated by p (see for instance [8, p. 323, Proposition (1.8)]). Assume $\mu_{p^\infty} \subset K'_\infty$. Then the kernel of the local norm residue symbol

$$(\cdot, K'_\infty/k') : (k')^\times \rightarrow \text{Gal}(K'_\infty/k')$$

is a subgroup of $\langle p \rangle$, whose index is finite. Let \mathfrak{Q} be a prime of k_∞ above $\bar{\mathfrak{p}}$. We write k'' for the completion of k at $\bar{\mathfrak{p}}$. For all $n \in \mathbb{N}$, we denote by k'_n (resp. k''_n) the completion of k_n at \mathfrak{P} (resp. \mathfrak{Q}). We set $k'_\infty := \bigcup_{i=0}^n k'_n$ and $k''_\infty := \bigcup_{i=0}^n k''_n$. Since $\bar{\mathfrak{p}}$ is finitely decomposed in k_∞/k , the extension k''_∞/k'' is infinite. But it is also unramified, and then its Galois group is topologically generated by $(p, k''_\infty/k'')$. By the product formula, and since k_∞/k is unramified outside of \mathfrak{p} , we have $(p, k''_\infty/k'')|_{k_\infty} = (p^{-1}, k'_\infty/k')|_{k_\infty}$, and we deduce that for all $n \in \mathbb{Z} \setminus \{0\}$, $(p^n, K'_\infty/k') \neq 1$. Hence $(\cdot, K'_\infty/k')$ is injective, which is absurd. \square

3 Elliptic units.

For L and L' two \mathbb{Z} -lattices of \mathbb{C} such that $L \subseteq L'$ and $[L' : L]$ is prime to 6, we denote by $z \mapsto \psi(z; L, L')$ the elliptic function defined in [13]. For \mathfrak{m} a nonzero proper ideal of \mathcal{O}_k , and \mathfrak{a} a nonzero ideal of \mathcal{O}_k prime to $6\mathfrak{m}$, G. Robert proved that $\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}) \in k(\mathfrak{m})$, where $k(\mathfrak{m})$ is the ray class field of k modulo \mathfrak{m} . More precisely, $\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}) \in \mathcal{O}_{k(\mathfrak{m})}^\times$ if \mathfrak{m} is divisible by at least two distinct primes, and if $\mathfrak{m} = \mathfrak{r}^n$ with \mathfrak{r} a prime ideal and $n \in \mathbb{N}^*$, then $\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})$ is a unit outside of the primes above \mathfrak{r} . For any maximal ideal \mathfrak{q} of \mathcal{O}_k , prime to \mathfrak{a} , by [12, Corollaire 1.3, (ii-1)] we have

$$N_{k(\mathfrak{m}\mathfrak{q})/k(\mathfrak{m})}(\psi(1; \mathfrak{m}\mathfrak{q}, \mathfrak{a}^{-1}\mathfrak{m}\mathfrak{q}))^{w_{\mathfrak{m}}/w_{\mathfrak{m}\mathfrak{q}}} = \begin{cases} \psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})^{1-(\mathfrak{q}, k(\mathfrak{m})/k)^{-1}} & \text{if } \mathfrak{q} \nmid \mathfrak{m}, \\ \psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}) & \text{if } \mathfrak{q} \mid \mathfrak{m}, \end{cases} \quad (3.1)$$

where $(\mathfrak{q}, k(\mathfrak{m})/k)$ is the Fröbenius of \mathfrak{q} in $k(\mathfrak{m})/k$, and $w_{\mathfrak{m}}$ is the number of roots of unity in k which are congruent to 1 modulo \mathfrak{m} . Moreover, by [12, Corollaire 1.3, (v-1)] we have

$$\psi(1; \mathfrak{m}\mathfrak{q}, \mathfrak{a}^{-1}\mathfrak{m}\mathfrak{q}) \equiv \psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})^{(\mathfrak{q}, k(\mathfrak{m})/k)} \pmod{(\mathfrak{q})_{\mathfrak{m}\mathfrak{q}}}, \quad (3.2)$$

where $(\mathfrak{q})_{\mathfrak{m}\mathfrak{q}}$ is the product of the prime ideals in $\mathcal{O}_{k(\mathfrak{m}\mathfrak{q})}$ above \mathfrak{q} .

Definition 3.1 Let $F \subseteq \mathbb{C}$ be a finite abelian extension of k , and write $\mu(F)$ for the group of roots of unity in F . We write Ψ_F for the $\mathbb{Z}[\text{Gal}(F/k)]$ -submodule of F^\times generated by $\mu(F)$ and by all the norms

$$N_{k(\mathfrak{m})/k(\mathfrak{m}) \cap F}(\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})),$$

where \mathfrak{m} is a nonzero proper ideal of \mathcal{O}_k and \mathfrak{a} is any nonzero ideal of \mathcal{O}_k prime to $6\mathfrak{m}$. Then, we define the group

$$C_F := \Psi_F \cap \mathcal{O}_F^\times.$$

Remark 3.1 For any $n \in \mathbb{N}$, \mathcal{U}_n is canonically identified to the pro- p -completion of the group of semi-local units U_n of K_n . Hence the natural inclusions $\mathcal{O}_{K_n}^\times \hookrightarrow U_n$ induce norm compatible canonical maps $\mathcal{E}_n \rightarrow \mathcal{U}_n$. The Leopoldt conjecture, which is known to be true for abelian extensions of k , states that this map is injective. Taking the projective limits, we obtain a natural injection $\mathcal{E}_\infty \hookrightarrow \mathcal{U}_\infty$.

Proposition 3.1 The $\mathbb{Z}_p[[T]]$ -module $\mathcal{U}_\infty/\mathcal{C}_\infty$ is finitely generated and torsion.

Proof. For all $n \in \mathbb{N}$, we let St_n be the group of Stark units defined in [6, Definition 3.2], and we set $\overline{St}_n := \mathbb{Z}_p \otimes_{\mathbb{Z}} St_n$ and $\overline{St}_\infty := \varprojlim \overline{St}_n$ (projective limit with respect to the norm maps). It is well known that Stark units can be constructed by means of elliptic units (for instance, see [9, Chapitre V, 4] for a precise statement). Then it is an easy matter to verify that $St_n \subseteq C_n$ for all $n \in \mathbb{N}$. Hence $\overline{St}_\infty \subseteq \mathcal{C}_\infty$, and we just have to show that $\mathcal{U}_\infty/\overline{St}_\infty$ is finitely generated and torsion. By [6, Theorem 3.2 and Proposition 2.1], we know that \overline{St}_∞ is torsion-free of rank $[K_0 : k]$ over $\mathbb{Z}_p[[T]]$. Then from [5, Theorem 25] and Remark 3.1, we deduce that $\mathcal{U}_\infty/\overline{St}_\infty$ is finitely generated and torsion over $\mathbb{Z}_p[[T]]$. \square

4 Euler systems.

Let us write A_k as a direct product of cyclic p -groups,

$$A_k = \langle \text{cl}(\mathfrak{p}_1) \rangle \times \cdots \times \langle \text{cl}(\mathfrak{p}_r) \rangle,$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are prime ideals of \mathcal{O}_k , prime to p , and $\text{cl}(\mathfrak{p}_i)$ is the class of \mathfrak{p}_i in $\text{Cl}(\mathcal{O}_k)$. For any $i \in \{1, \dots, r\}$, let p^{Ri} be the order of $\langle \text{cl}(\mathfrak{p}_i) \rangle$, and we choose $\alpha_i \in \mathcal{O}_k$ be such that $\alpha_i \mathcal{O}_k = \mathfrak{p}_i^{p^{\text{Ri}}}$. Let $\text{R} := \sum_{i=1}^r \text{R}_i$ and let $\text{M} \neq 1$ be a power of p , such that $p^{\text{R}} = \#(A_k) \leq \text{M}$.

Let $\omega := 1$ if $p \neq 2$, and $\omega := -1$ if $p = 2$.

We denote by \mathcal{L}_F the set of maximal ideals ℓ of \mathcal{O}_k such that ℓ splits completely in $F(\mu_{\text{M}}, \sqrt[\text{M}]{\omega}, \sqrt[\text{M}]{\alpha_1}, \dots, \sqrt[\text{M}]{\alpha_r})/k$, and such that $\ell \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. We denote by \mathcal{S}_F the set of squarefree ideals of \mathcal{O}_k whose prime divisors belongs to \mathcal{L}_F . As in [11, Lemma 3.1], we define for each $\ell \in \mathcal{L}_F$ a cyclic subextension $F(\ell)$ of $k(\ell)F$, of degree M , which is totally ramified above ℓ and unramified anywhere else. For $\mathfrak{m} := \ell_1 \cdots \ell_n$ an ideal in \mathcal{S}_F , we define $F(\mathfrak{m}) := F(\ell_1) \cdots F(\ell_n)$, the compositum of the fields $F(\ell_i)$.

For any ideal $\mathfrak{m} \neq 0$ of \mathcal{O}_k , we denote by $\mathcal{S}_F(\mathfrak{m})$ the set of ideals in \mathcal{S}_F which are prime to \mathfrak{m} . We denote by $\mathcal{U}_F(\mathfrak{m})$ the set of maps $\epsilon : \mathcal{S}_F(\mathfrak{m}) \rightarrow (k^{ab})^\times$ satisfying the conditions (a) to (d) below.

$$(a) \quad \epsilon(\mathfrak{a}) \in F(\mathfrak{a})^\times \text{ for all } \mathfrak{a} \in \mathcal{S}_F(\mathfrak{m}).$$

$$(b) \quad \epsilon(\mathfrak{a}) \in \mathcal{O}_{F(\mathfrak{a})}^\times \text{ if } \mathfrak{a} \neq (1).$$

$$(c) \quad N_{F(\mathfrak{a}\ell)/F(\mathfrak{a})}(\epsilon(\mathfrak{a}\ell)) = \epsilon(\mathfrak{a})^{(\ell, F(\mathfrak{a})/k)-1} \text{ for all } \mathfrak{a} \in \mathcal{S}_F(\mathfrak{m}) \text{ and all } \ell \in \mathcal{L}_F \text{ which is prime to } \mathfrak{m}\mathfrak{a}.$$

$$(d) \quad \epsilon(\mathfrak{a}\ell) \equiv \epsilon(\mathfrak{a})^{(N(\ell)-1)/\text{M}} \text{ modulo all prime ideals of } \mathcal{O}_{F(\mathfrak{a})} \text{ above } \ell.$$

Remark 4.1 Let $\mathcal{U} := \cup \mathcal{U}_F(\mathfrak{m})$, where the union is over all nonzero ideal \mathfrak{m} of \mathcal{O}_k . Then for any $u \in C(F)$, there exists $\epsilon \in \mathcal{U}$ such that $\epsilon(1) = u$ (see [15, Proposition 1.2]).

For any ideal $\mathfrak{a} \neq (0)$ of \mathcal{O}_k and any $\epsilon \in \mathcal{U}_F(\mathfrak{a})$, we denote $\kappa_\epsilon : \mathcal{S}_F(\mathfrak{a}) \rightarrow F^\times / (F^\times)^\text{M}$ the map defined as in [15, Proposition 2.2]. For $\ell \in \mathcal{L}_F$, we let $\mathcal{I}_{F,\ell} := \bigoplus_{\lambda|\ell} \mathbb{Z}\lambda$ be the free \mathbb{Z} -module generated by the prime ideals of \mathcal{O}_F lying above ℓ . For any $x \in F^\times$, we denote by $(x)_\ell \in \mathcal{I}_{F,\ell}$ and $[x]_\ell \in \mathcal{I}_{F,\ell}/\text{M}\mathcal{I}_{F,\ell}$ the projections of the fractional ideal $(x) := x\mathcal{O}_K$. Consider the map

$$\theta_\ell : F(\ell)^\times \longrightarrow (\mathcal{O}_F/\ell\mathcal{O}_F)^\times / ((\mathcal{O}_F/\ell\mathcal{O}_F)^\times)^\text{M},$$

which associates to z the sum $\bigoplus_{\lambda|\ell} z_\lambda$ such that the image of $z^{1-\sigma_\ell}$ in $(\mathcal{O}_F/\lambda)^\times$ is equal to $(z_\lambda)^{(N(\ell)-1)/\text{M}}$. As in [15, Proposition 2.3], there exists a unique $\text{Gal}(F/k)$ -equivariant isomorphism

$$\varphi_\ell : (\mathcal{O}_F/\ell\mathcal{O}_F)^\times / ((\mathcal{O}_F/\ell\mathcal{O}_F)^\times)^\text{M} \rightarrow \mathcal{I}_{F,\ell}/\text{M}\mathcal{I}_{F,\ell},$$

satisfying the relation $(\varphi_\ell \circ \theta_\ell)(x) = [N_{F(\ell)/F}(x)]_\ell$. For $x \in F^\times$, we can choose $y \in F(\ell)^\times$ such that xy^M is a unit at the prime ideals of $\mathcal{O}_{F(\ell)}$ above ℓ . We denote by $\{xy^\text{M}\}$ the

class of $xy^{\mathbf{m}}$ in $(\mathcal{O}_{F(\ell)}/\ell')^{\times} / ((\mathcal{O}_{F(\ell)}/\ell')^{\times})^{\mathbf{m}} \simeq (\mathcal{O}_F/\ell\mathcal{O}_F)^{\times} / ((\mathcal{O}_F/\ell\mathcal{O}_F)^{\times})^{\mathbf{m}}$, where ℓ' is the product of the prime ideals of $\mathcal{O}_{F(\ell)}$ above ℓ . Then we set $\varphi_{\ell}(x) := \varphi_{\ell}(\{xy^{\mathbf{m}}\})$, which does not depend on the choice of y .

Then as in [15, Proposition 2.4], for any ideal $\mathfrak{a} \neq (0)$ of \mathcal{O}_k , any $\epsilon \in \mathcal{U}_F(\mathfrak{a})$, any $\mathfrak{m} \in \mathcal{S}_F(\mathfrak{a})$ with $\mathfrak{m} \neq (1)$, and any maximal ℓ of \mathcal{O}_k , we have

$$[\kappa_{\epsilon}(\mathfrak{m})]_{\ell} = \begin{cases} 0 & \text{if } \ell \nmid \mathfrak{m}, \\ \varphi_{\ell}(\kappa_{\epsilon}(\mathfrak{m}\ell^{-1})) & \text{if } \ell \mid \mathfrak{m}. \end{cases} \quad (4.1)$$

For any $x \in F^{\times}$, we denote by $\langle x \rangle_{\mathbf{m}}$ the class of x in $F^{\times} / (F^{\times})^{\mathbf{m}}$. For any $n \in \mathbb{N}$ we denote by μ_n the group of n -th roots of unity in \mathbb{C} . We set $\mu_{p^{\infty}} := \bigcup_{n=0}^{\infty} \mu_{p^n}$. For any extension $L \subseteq F$ of k and any maximal ideal \mathfrak{q} of \mathcal{O}_L , we denote by $v_{\mathfrak{q}}$ the normalized valuation at \mathfrak{q} , and by $\bar{v}_{\mathfrak{q}} : L^{\times} / (L^{\times})^{\mathbf{m}} \rightarrow \mathbb{Z}/\mathbf{m}\mathbb{Z}$ the map defined from $v_{\mathfrak{q}}$ by taking the quotient.

The following theorem is a classical step in the Euler system machinery. The first versions are due to Rubin (see [15, Theorem 3.1]), and to Greither for abelian extensions over \mathbb{Q} (see [4, Theorem 3.7]). We follow the proof of Bley (see [1, Theorem 3.4]), with slight modifications to cover the case $p \nmid \#(\text{Cl}(\mathcal{O}_k))$.

Theorem 4.1 *Let \mathfrak{f} be the conductor of F/k , and set $c := v_{\bar{\mathfrak{p}}}(\mathfrak{f})$. We set $G_F := \text{Gal}(F/k)$. Assume that we are given an ideal class $\mathfrak{c} \in A_F$, a finite $\mathbb{Z}[G_F]$ -submodule W of $F^{\times} / (F^{\times})^{\mathbf{m}}$, and a G_F -morphism $\Psi : W \rightarrow \mathbb{Z}/\mathbf{m}\mathbb{Z}[G_F]$.*

Assume that for all $w \in W$, all $i \in \{1, \dots, r\}$, and all prime \mathfrak{q} of F above \mathfrak{p}_i , $\bar{v}_{\mathfrak{q}}(w) = 0$. Assume also that for any $i = 1, \dots, r$, \mathfrak{p}_i is unramified in F/k . Let m be a positive integer divisible by p^{2c+1} . Then there are infinitely many maximal ideals λ of \mathcal{O}_F such that

- (i) $\text{cl}_p(\lambda) = \mathfrak{c}^m$.*
- (ii) $\ell := \lambda \cap \mathcal{O}_k$ belongs to \mathcal{L}_F .*
- (iii) For all $w \in W$, $[w]_{\ell} = 0$.*
- (iv) There exists $u \in (\mathbb{Z}/\mathbf{m}\mathbb{Z})^{\times}$, such that for all $w \in W$, $\varphi_{\ell}(w) = u p^{3c+r+4} \Psi(w) \lambda$.*

Proof. Let H_F be the Hilbert p -class field of F . Let

$$F_i := \begin{cases} F(\mu_{\mathbf{m}}) & \text{if } i = 0, \\ F_{i-1}(\sqrt[m]{\alpha_i}) & \text{if } 1 \leq i \leq r. \end{cases}$$

Exactly as in [1, proof of Theorem 3.4], one can prove the following claims.

Claim (A) $[H_F \cap F(\mu_{p^{\infty}}) : F] \leq p^c$.

Claim (B) $\text{Gal}\left(H_F \cap F_r\left(\sqrt[m]{\omega}, \sqrt[m]{W}\right) / F\right)$ is annihilated by p^{2c+1} .

Claim (C) The cokernel of the canonical map from Kummer theory

$$\mathfrak{K} : \text{Gal}\left(F_0\left(\sqrt[m]{W}\right) / F_0\right) \hookrightarrow \text{Hom}(W, \mu_{\mathbf{m}})$$

is annihilated by p^{c+2} .

Let us remark that $F_{i-1}\left(\sqrt[m]{W}\right) / F_{i-1}$ is unramified at \mathfrak{p}_i since by hypothesis $\mathbf{m} \mid v_{\mathfrak{p}_i}(w)$ for all $\langle w \rangle_{\mathbf{m}} \in W$. On the other hand $[F_i : F_{i-1}]$ divides \mathbf{m} and the ramification index of \mathfrak{p}_i in F_i / F_{i-1} is at least $\mathbf{m} p^{-R_i}$. Therefore

$$p^{R_i} \text{ annihilates } \text{Gal}\left(F_{i-1}\left(\sqrt[m]{W}\right) \cap F_i / F_{i-1}\right). \quad (4.2)$$

Let $L_i := F_0 \left(\sqrt[m]{W} \right) \cap F_i$. As $L_i \cap F_{i-1} = L_{i-1}$ we have

$$\text{Gal}(L_i/L_{i-1}) \simeq \text{Gal}(L_i F_{i-1}/F_{i-1}). \quad (4.3)$$

Since $\text{Gal}(L_i F_{i-1}/F_{i-1})$ is a quotient of $\text{Gal}\left(F_{i-1} \left(\sqrt[m]{W} \right) \cap F_i/F_{i-1}\right)$, this implies that $p^{\mathbf{r}_i}$ annihilates $\text{Gal}(L_i/L_{i-1})$ thanks to (4.2). In particular $p^{\mathbf{r}}$ annihilates $\text{Gal}(L_r/F_0)$, and we deduce Claim (D) below.

Claim (D) $\text{Gal}\left(F_0 \left(\sqrt[m]{W} \right) \cap F_r \left(\sqrt[m]{W} \right) / F_0\right)$ is annihilated by $p^{\mathbf{r}+1}$.

Let ζ be a primitive m -root of unity, and $\iota : \mathbb{Z}/m\mathbb{Z}[G_F] \rightarrow \mu_m$ be the group morphism such that $\iota(\sigma) = 0$ for $\sigma \in G_F \setminus \{1\}$ and $\iota(1) = \zeta$. Combining Claim (C) and Claim (D), one may find $\alpha \in \text{Gal}\left(F_r \left(\sqrt[m]{W}, \sqrt[m]{W} \right) / F_0\right)$ such that

$$\alpha|_{F_r(\sqrt[m]{W})} = 1 \quad \text{and} \quad \mathfrak{K}\left(\alpha|_{F_0(\sqrt[m]{W})}\right) = (\iota \circ \Psi)^{p^{\mathbf{r}+c+3}}. \quad (4.4)$$

From Claim (B), we may choose $\beta \in \text{Gal}\left(H_F F_r \left(\sqrt[m]{W}, \sqrt[m]{W} \right) / F\right)$ such that

$$\beta|_{F_r(\sqrt[m]{W}, \sqrt[m]{W})} = \alpha^{p^{2c+1}} \quad \text{and} \quad \beta|_{H_F} = \mathfrak{c}^m. \quad (4.5)$$

Now, from (4.4) we see that $\beta \in \text{Gal}\left(H_F F_r \left(\sqrt[m]{W}, \sqrt[m]{W} \right) / F_r \left(\sqrt[m]{W} \right)\right)$.

By the Čebotarev density theorem, we can find infinitely many primes λ in \mathcal{O}_F , of absolute degree 1, prime to $\prod_{i=1}^r \mathfrak{p}_i$, such that $\lambda \cap \mathcal{O}_k$ is unramified in $H_F F_r \left(\sqrt[m]{W}, \sqrt[m]{W} \right) / k$, and such that the conjugacy class of β in $\text{Gal}\left(H_F F_r \left(\sqrt[m]{W}, \sqrt[m]{W} \right) / F\right)$ is the Fröbenius of λ . Then condition (i) of Theorem 4.1 holds as a consequence of the general properties of the Fröbenius. The condition (ii) is also satisfied since β is the identity on $F_r \left(\sqrt[m]{W} \right)$. Let $w \in W$. Then for any prime λ' of $\mathcal{O}_{F_0(\sqrt[m]{W})}$ above λ , we have $\bar{v}_\lambda(w) = \bar{v}_{\lambda'}(w) = m\bar{v}_{\lambda'}(\sqrt[m]{w}) = 0$, and condition (iii) follows. Condition (iv) is proved as in the proof of [15, Theorem 8.1,(iii)]. \square

For any $\mathbb{Z}_p[G]$ -module M and any $m \in M$, we denote by m_χ the canonical image of m in M_χ .

Lemma 4.1 *Let \mathcal{G} be a subgroup of G_F , and let χ be an irreducible \mathbb{C}_p character of \mathcal{G} . Let $\ell_1, \dots, \ell_i \in \mathcal{L}_F$, and for any $j = 1, \dots, i$, let λ_j be a prime of \mathcal{O}_F above ℓ_j , and let $\text{cl}_p(\lambda_j)$ be the image of λ_j in A_F . Let $x \in F^\times$ be such that $v_{\mathfrak{q}}(x) \in m\mathbb{Z}$ for any prime \mathfrak{q} of \mathcal{O}_F which is prime to $\ell_1 \cdots \ell_i$. Let W be the $\mathbb{Z}_p[G_F]$ -span of the image of x in $F^\times / (F^\times)^m$, and let L be the $\mathbb{Z}_p[G_F]$ -module of A_F generated by $\text{cl}_p(\lambda_1), \dots, \text{cl}_p(\lambda_{i-1})$. Assume that there are $Z, g, \eta \in \mathbb{Z}_p[G_F]$ such that*

(i) $Z \cdot \text{Ann}_{\mathbb{Z}_p[G_F]_\chi} \left([\text{cl}_p(\lambda_i)]_{L, \chi} \right) \subseteq g\mathbb{Z}_p[G_F]_\chi$, where $\text{Ann}_{\mathbb{Z}_p[G_F]_\chi} \left([\text{cl}_p(\lambda_i)]_{L, \chi} \right)$ is the annihilator of the image $[\text{cl}_p(\lambda_i)]_{L, \chi}$ of $\text{cl}_p(\lambda_i)$ in $(A_F/L)_\chi$.

(ii) $\mathbb{Z}_p[G_F]_\chi / g\mathbb{Z}_p[G_F]_\chi$ is finite.

(iii) $\# \left(\eta \left((\mathcal{I}_{F,\ell_i} / M\mathcal{I}_{F,\ell_i}) / W' \right)_\chi \right) \# (A_{F,\chi}) \leq M$, where W' is the image of W in $\mathcal{I}_{F,\ell_i} / M\mathcal{I}_{F,\ell_i}$ through $w \mapsto [w]_{\ell_i}$.

Then, there exists a morphism of $\mathbb{Z}_p[G_F]$ -modules

$$\Psi : W_\chi \rightarrow \mathbb{Z}/M\mathbb{Z}[G_F]_\chi$$

such that

$$g\Psi \left(\langle x \rangle_{M,\chi} \right) \lambda_{i,\chi} = Z\eta[x]_{\ell_i,\chi}.$$

Proof. We refer the reader to [4, Lemma 3.12]. \square

5 The ideal class group.

Proposition 5.1 *The projective limit A_∞ is a finitely generated torsion $\mathbb{Z}_p[[T]]$ -module. Moreover, for all $n \in \mathbb{N}$, A_{∞,Γ_n} and $A_\infty^{\Gamma_n}$ are finite.*

Proof. We refer the reader to [14, Proof of Theorem 1.4]. \square

Let $\tau : A_{\infty,\chi} \rightarrow \bigoplus_{j=1}^s \Lambda_\chi / P_j$ be a pseudo-isomorphism of Λ_χ -modules, where P_1, \dots, P_s are nonzero polynomials in Λ_χ .

Let $M_\infty := \varprojlim (M_n)$ be a $\mathbb{Z}_p[[T]]$ -module, projective limit of $\mathbb{Z}_p[\Gamma/\Gamma_n]$ -modules M_n . For all $n \in \mathbb{N}$, we denote by $\text{Ker}_n M_\infty$ and $\text{Cok}_n M_\infty$ the respective kernel and cokernel of the canonical map $M_{\Gamma_n} \rightarrow M_n$.

Lemma 5.1 *There is $c_3 \in \mathbb{N}$, and for all $n \in \mathbb{N}$, there is a morphism of Λ_χ -modules*

$$\tau_n : A_{n,\chi} \rightarrow \bigoplus_{j=1}^s \Lambda_\chi / (P_j, 1 - \gamma_n)$$

such that $\text{Cok}(\tau_n)$ is annihilated by p^{c_3} .

Proof. Let $m \in \mathbb{N}$ be such that K_∞/K_m is totally ramified above \mathfrak{p} . By [19, Lemma 13.15], there is a $\mathbb{Z}_p[[\Gamma_m]]$ -submodule Y of A_∞ such that for all $n \geq m$, the canonical map $A_\infty \rightarrow A_n$ induces an isomorphism

$$A_\infty / \nu_{m,n} Y \xrightarrow{\sim} A_n,$$

where $\nu_{m,n} \in \mathbb{Z}_p[[T]]$ is defined by $\nu_{m,n} := (1 - \gamma_n) / (1 - \gamma_m)$. Therefore for all $n \geq m$, we have $\text{Cok}_n A_\infty = 0$ and

$$\text{Ker}_n A_\infty \simeq \nu_{m,n} Y / (1 - \gamma_n) A_\infty. \quad (5.1)$$

Multiplication by $\nu_{m,n}$ induces a surjection

$$Y / (1 - \gamma_m) A_\infty \longrightarrow \nu_{m,n} Y / (1 - \gamma_n) A_\infty, \quad (5.2)$$

from which we deduce that for all $n \geq m$, $\text{Ker}_n A_\infty$ is a quotient of $\text{Ker}_m A_\infty$. Since $\text{Ker}_n A_\infty$ is finite, by Proposition 5.1 we see that the orders of $\text{Ker}_n A_\infty$ and $\text{Cok}_n A_\infty$ are bounded independantly of n . We choose $\alpha \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, p^α annihilates

$\text{Ker}_n A_\infty$ and $\text{Cok}_n A_\infty$. On the other hand, since $\text{Cok}_n A_\infty = 0$, we have the exact sequence below for any $n \geq m$,

$$(\text{Ker}_n A_\infty)_\chi \longrightarrow (A_\infty)_{\Gamma_n, \chi} \longrightarrow A_{n, \chi} \longrightarrow 0.$$

This shows that p^α annihilates $\text{Ker}((A_{\infty, \chi})_{\Gamma_n} \rightarrow A_{n, \chi})$, for all $n \geq m$. Moreover for all $n \in \mathbb{N}$, $(A_{\infty, \chi})_{\Gamma_n} \simeq (A_{\infty, \Gamma_n})_\chi$ is finite from Proposition 5.1. Thus we may choose α such that p^α also annihilates $\text{Ker}((A_{\infty, \chi})_{\Gamma_n} \rightarrow A_{n, \chi})$ for all $n \in \mathbb{N}$. Then choose $\beta \in \mathbb{N}$ such that p^β annihilates $\text{Cok}(\tau)$, and set $c_3 := 2\alpha + \beta$. Let $\bar{\tau}_n : (A_{\infty, \chi})_{\Gamma_n} \rightarrow \bigoplus_{j=1}^s \Lambda_\chi / (P_j, 1 - \gamma_n)$ be the morphism of Λ_χ -modules defined from τ by taking the quotients, and set

$$\tau_n : A_{n, \chi} \longrightarrow \bigoplus_{j=1}^s \Lambda_\chi / (P_j, 1 - \gamma_n), \quad x \longmapsto p^\alpha \bar{\tau}_n(y),$$

where $y \in (A_{\infty, \chi})_{\Gamma_n}$ is such that its image in $A_{n, \chi}$ is $p^\alpha x$. It is straightforward that τ_n is well-defined, and that the condition of the lemma is satisfied. \square

6 Global units.

Let us fix χ an irreducible \mathbb{C}_p character of G , and let $\psi : G \rightarrow \mathbb{Z}_p$ the irreducible \mathbb{Q}_p character of G such that $\chi|\psi$. Also we denote by \mathbf{u}_χ a fixed uniformizer of $\mathbb{Z}_p(\chi)$.

Lemma 6.1 *There is a finite set I , a family $(n_i)_{i \in I} \in \mathbb{N}^I$, and a pseudo-isomorphism of Λ_χ -modules:*

$$\Theta : \mathcal{E}_{\infty, \chi} \rightarrow \Lambda_\chi \oplus \bigoplus_{i \in I} (\Lambda_\chi / \mathbf{u}_\chi^{n_i}). \quad (6.1)$$

Proof. From [5, Theorem 25] and Remark 3.1, we know that \mathcal{E}_∞ is finitely generated over $\mathbb{Z}_p[[T]]$. We denote by $\Lambda(\psi)$ the principal ring $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda)$. From the tautological exact sequence $0 \rightarrow \mathcal{E}_\infty \rightarrow \mathcal{U}_\infty \rightarrow \mathcal{U}_\infty / \mathcal{E}_\infty \rightarrow 0$ we deduce the following exact sequence of $\Lambda(\psi)$ -modules

$$0 \rightarrow e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{E}_\infty) \rightarrow e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}_\infty) \rightarrow e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}_\infty / \mathcal{E}_\infty) \rightarrow 0. \quad (6.2)$$

But we know that $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}_\infty)$ is free of rank 1 over $\Lambda(\psi)$, thanks to Proposition 2.1. Moreover, $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{U}_\infty / \mathcal{E}_\infty)$ is $\Lambda(\psi)$ -torsion by Proposition 3.1. As $\Lambda(\psi)$ is a principal ring, we see from (6.2) that $e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{E}_\infty)$ is free of rank 1 over $\Lambda(\psi)$. The isomorphisms

$$e_\psi(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{E}_\infty) \simeq \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{E}_{\infty, \chi} \quad \text{and} \quad \Lambda(\psi) \simeq \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_\chi$$

imply that the Λ_χ -torsion of $\mathcal{E}_{\infty, \chi}$ is annihilated by some power of p , and the Λ_χ -rank of $\mathcal{E}_{\infty, \chi}$ is 1. \square

Lemma 6.2 *There is $(c_0, n_0) \in \mathbb{N}^2$ such that for all $n \in \mathbb{N}$, $p^{c_0}(\gamma_{n_0} - 1)$ annihilates $\text{Cok}_n \mathcal{E}_\infty$ and $\text{Ker}_n \mathcal{E}_\infty$.*

Proof. The proof is similar to [10, Corollary 3.9]. \square

For every $n \in \mathbb{N}$, the projection $\mathcal{E}_\infty \rightarrow \mathcal{E}_n$ induces a natural map $\pi_{n, \chi} : (\mathcal{E}_{\infty, \chi})_{\Gamma_n} \rightarrow \mathcal{E}_{n, \chi}$.

Lemma 6.3 For all $n \in \mathbb{N}$, $p^{2c_0}(\gamma_{n_0} - 1)^2$ annihilates $\text{Cok}(\pi_{n,\chi})$ and $\text{Ker}(\pi_{n,\chi})$.

Proof. Let $n \in \mathbb{N}$ and let $\mathcal{T} := \text{Tor}_{\mathbb{Z}_p[G]}^1(\text{Cok}_n \mathcal{E}_\infty, \mathbb{Z}_p(\chi))$. We denote by $\tilde{\mathcal{E}}_n$ the image of \mathcal{E}_∞ in \mathcal{E}_n , and we write $\tilde{\pi}_{n,\chi} : (\mathcal{E}_{\infty,\chi})_{\Gamma_n} \rightarrow \tilde{\mathcal{E}}_{n,\chi}$. From the following commutative exact diagram,

$$\begin{array}{ccccccc}
 & & & & \mathcal{T} & & \\
 & & & & \downarrow & & \\
 (\text{Ker}_n \mathcal{E}_\infty)_\chi & \longrightarrow & (\mathcal{E}_{\infty,\chi})_{\Gamma_n} & \longrightarrow & \tilde{\mathcal{E}}_{n,\chi} & \longrightarrow & 0 \\
 \downarrow & & \downarrow \pi_{n,\chi} & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{E}_{n,\chi} & \xlongequal{\quad} & \mathcal{E}_{n,\chi} & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & (\text{Cok}_n \mathcal{E}_\infty)_\chi & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array} ,$$

we deduce an exact sequence

$$(\text{Ker}_n \mathcal{E}_\infty)_\chi \longrightarrow \text{Ker}(\pi_{n,\chi}) \longrightarrow \tilde{\mathcal{T}} \longrightarrow 0, \quad (6.3)$$

where $\tilde{\mathcal{T}}$ is the image of \mathcal{T} in $\tilde{\mathcal{E}}_{n,\chi}$. By Lemma 6.2, we know that $p^{c_0}(\gamma_{n_0} - 1)$ annihilates \mathcal{T} and $(\text{Ker}_n \mathcal{E}_\infty)_\chi$. Therefore (6.3) implies the desired result for $\text{Ker}(\pi_{n,\chi})$. On the other hand, since $\text{Cok}(\pi_{n,\chi}) \simeq (\text{Cok}_n \mathcal{E}_\infty)_\chi$ the lemma is entirely proved. \square

Let $\text{pr} : \Lambda_\chi \oplus \bigoplus_{i \in I} (\Lambda_\chi / \mathfrak{u}_\chi^{n_i}) \rightarrow \Lambda_\chi$ be the canonical projection, and for every n let

$$\Theta_n : (\mathcal{E}_{\infty,\chi})_{\Gamma_n} \longrightarrow \mathbb{Z}_p(\chi) [\Gamma / \Gamma_n]$$

be the map obtained from $\text{pr} \circ \Theta$ by taking the quotients. By Remark 3.1, $\mathcal{E}_\infty / \mathcal{C}_\infty$ is a submodule of $\mathcal{U}_\infty / \mathcal{C}_\infty$, hence is finitely generated and torsion over $\mathbb{Z}_p[[T]]$ by Proposition 3.1. We denote by h_χ a generator of $\text{char}(\mathcal{E}_\infty / \mathcal{C}_\infty)_\chi$. The following lemma is the analogue of [1, Lemma 3.5].

Lemma 6.4 Let $n \in \mathbb{N}$. Then there is a map

$$\vartheta_{n,\chi} : \mathcal{E}_{n,\chi} \rightarrow \mathbb{Z}_p(\chi) [\Gamma / \Gamma_n], x \mapsto (\gamma_{n_0} - 1)^2 p^{2c_0} \Theta_n(\underline{x}),$$

where $\underline{x} \in \mathcal{E}_{\infty,\chi}$ is such that $(\gamma_{n_0} - 1)^2 p^{2c_0} x = \pi_{n,\chi}(\underline{x})$. Then there are $(\nu, c_1, c_2) \in \mathbb{N}^3$ and $h'_\chi \in \Lambda_\chi$ such that

- (i) $h'_\chi | h_\chi$ in Λ_χ .
- (ii) For all $n \in \mathbb{N}$, h'_χ is prime to $1 - \gamma_n$ in Λ_χ .
- (iii) For all $n \in \mathbb{N}$, $(\gamma_\nu - 1)^{c_1} p^{c_2} h'_\chi \mathbb{Z}_p(\chi) [\Gamma / \Gamma_n] \subseteq \vartheta_{n,\chi}(\text{Im}(\mathcal{C}_{n,\chi}))$, where $\text{Im}(\mathcal{C}_{n,\chi})$ is the image of $\mathcal{C}_{n,\chi}$ in $\mathcal{E}_{n,\chi}$.

Proof. One may use Lemma 6.3 to verify that $\vartheta_{n,\chi}$ is well-defined. We leave the details to the reader. The module $h_\chi \cdot (\mathcal{E}_\infty / \mathcal{C}_\infty)_\chi$ is finite, and so is $h_\chi \cdot (\Theta(\mathcal{E}_{\infty,\chi}) / \Theta(\text{Im}(\mathcal{C}_{\infty,\chi})))$.

Since $\text{Cok}(\text{pr} \circ \Theta)$ is also finite, we can choose $m \in \mathbb{N}$ such that $p^m h_\chi \in \text{pr} \circ \Theta(\text{Im}(\mathcal{C}_{\infty, \chi}))$. Let $z \in \text{Im}(\mathcal{C}_{\infty, \chi})$ be such that $p^m h_\chi = \text{pr} \circ \Theta(z)$. Then in $\mathbb{Z}_p(\chi) [\Gamma/\Gamma_n]$ we have

$$p^{m+4c_0} (\gamma_{n_0} - 1)^4 h_\chi = p^{4c_0} (\gamma_{n_0} - 1)^4 \Theta_n(z) = \vartheta_{n, \chi}(\pi_{n, \chi}(z)). \quad (6.4)$$

Let \mathcal{Q} be the set of prime ideals \mathfrak{q} of Λ_χ of height 1, and for any $\mathfrak{q} \in \mathcal{Q}$, let $P_{\mathfrak{q}}$ be a generator of \mathfrak{q} . Since Λ_χ is factorial, there is a unit $u \in \Lambda_\chi^\times$ and a family $(n_{\mathfrak{q}})_{\mathfrak{q} \in \mathcal{Q}} \in \mathbb{N}^{\mathcal{Q}}$ with finite support such that $h_\chi = u \prod_{\mathfrak{q} \in \mathcal{Q}} P_{\mathfrak{q}}^{n_{\mathfrak{q}}}$. We set $h'_\chi := \prod_{\mathfrak{q} \in \mathcal{Q}'} P_{\mathfrak{q}}^{n_{\mathfrak{q}}}$, where \mathcal{Q}' is the set of all $\mathfrak{q} \in \mathcal{Q}$ such that \mathfrak{q} is prime to $1 - \gamma_n$, for all $n \in \mathbb{N}$. Since $1 - \gamma_n$ divides $1 - \gamma_{n+1}$ for all $n \in \mathbb{N}$, we can choose $\nu \in \mathbb{N}$ and $c_1 \in \mathbb{N}$ such that $(\gamma_{n_0} - 1)^4 h_\chi$ divides $(\gamma_\nu - 1)^{c_1} h'_\chi$. Also, we set $c_2 := m + 4c_0$. Then the lemma follows from (6.4). \square

7 Proof of Theorem 1.1.

This section is devoted to the proof of Theorem 1.1. We choose c_0 and n_0 as in Lemma 6.2, $c_1 \geq 2$, c_2 and ν as in Lemma 6.4, and c_3 as in Lemma 5.1. Let us define

$$d := 3v_{\mathfrak{p}}(\mathfrak{f}) + R + 4 \quad \text{and} \quad \Delta_i := p^{(i-2)(c_3+2d)+d+c_2} [K_0 : k]^{i-1} \quad (i \geq 2),$$

where \mathfrak{f} is a nonzero ideal of \mathcal{O}_k such that $K_\infty \subseteq \bigcup_{n=0}^{\infty} k(\mathfrak{f}\mathfrak{p}^n)$. Let $n \in \mathbb{N}$. Since h'_χ is prime to $1 - \gamma_n$, the factor group

$$\mathbb{Z}_p[G_n]_\chi / \Delta_{s+1} h'_\chi \mathbb{Z}_p[G_n]_\chi \simeq \Lambda_\chi / ((1 - \gamma_n) \Lambda_\chi + \Delta_{s+1} h'_\chi \Lambda_\chi)$$

is finite. Let M be a power of p such that

$$\#A_k \#A_{n, \chi} \#(\mathbb{Z}_p[G_n]_\chi / \Delta_{s+1} h'_\chi \mathbb{Z}_p[G_n]_\chi) \leq M. \quad (7.1)$$

Let us also introduce the following notation. If λ is a maximal ideal of \mathcal{O}_{K_n} such that $\ell := \lambda \cap \mathcal{O}_k \in \mathcal{L}_{K_n}$, then we denote by ω_λ and $\bar{\omega}_\lambda$ the maps

$$\omega_\lambda : K_n^\times \longrightarrow \mathbb{Z}_p[G_n], \quad \text{such that} \quad \omega_\lambda(x)\lambda = (x)_\ell,$$

and

$$\bar{\omega}_\lambda : K_n^\times / (K_n^\times)^M \longrightarrow (\mathbb{Z}/M\mathbb{Z})[G_n], \quad \text{such that} \quad \bar{\omega}_\lambda(\langle x \rangle_M) \lambda = [x]_\ell.$$

We know by Lemma 5.1 that for every $j \in \{1, \dots, s\}$, there is a class $\mathfrak{c}_j \in A_n$ such that

$$\tau_n(\mathfrak{c}_{j, \chi}) = (0, \dots, 0, p^{c_3}, 0, \dots, 0),$$

where p^{c_3} is at the j -th place. We recall that $\mathfrak{c}_{j, \chi}$ is the image of \mathfrak{c}_j in $A_{n, \chi}$. We also choose arbitrarily one more class $\mathfrak{c}_{s+1} \in A_n$. By the above Lemma 6.4 (iii), there is $\xi \in C_n$ such that

$$\vartheta_{n, \chi}(\xi') = (\gamma_\nu - 1)^{c_1} p^{c_2} h'_\chi \quad \text{in} \quad (\mathbb{Z}/M\mathbb{Z}[G_n])_\chi, \quad (7.2)$$

where ξ' is the image of ξ in $\text{Im}(\mathcal{C}_{n, \chi})$. Let us now fix an ideal \mathfrak{m} of \mathcal{O}_k and $\varepsilon \in \mathcal{U}_{K_n}(\mathfrak{m})$ such that $\kappa_\varepsilon(1) = \xi$. This is possible thanks to Lemma 4.1. The main step is to define recursively maximal ideals $\lambda_1, \dots, \lambda_{s+1}$ of \mathcal{O}_{K_n} and ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_{s+1}$ of \mathcal{O}_k such that

(a) $\ell_i := \lambda_i \cap \mathcal{O}_k$ belongs to \mathcal{L}_{K_n} for all $i = 1, \dots, s+1$.

(b) $\text{cl}_p(\lambda_i) = \mathfrak{c}_i^{p^d}$ for all $i = 1, \dots, s+1$.

(c) $\mathfrak{a}_i := \ell_1 \cdots \ell_i$.

(d) $\bar{\omega}_{\lambda_1}(\kappa_\varepsilon(\ell_1))_\chi = u_1 p^{c_2+d}[K_0 : k](\gamma_\nu - 1)^{c_1} h'_\chi$ in $(\mathbb{Z}/M\mathbb{Z}[G_n])_\chi$, for some $u_1 \in (\mathbb{Z}/M\mathbb{Z})^\times$.

(e) For every $i \in \{2, \dots, s+1\}$ there is $u_i \in (\mathbb{Z}/M\mathbb{Z})^\times$ such that

$$P_{i-1} \bar{\omega}_{\lambda_i}(\kappa_\varepsilon(\mathfrak{a}_i)) = u_i p^{c_3+2d}[K_0 : k](\gamma_\nu - 1)^{c_1^{i-1}} \bar{\omega}_{\lambda_{i-1}}(\kappa_\varepsilon(\mathfrak{a}_{i-1})).$$

Since we will use Theorem 4.1 for $F := K_n$, we assume from now until the end of this paper that

the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are unramified in K_∞/k .

Let us consider the map $\varpi \circ \vartheta_{n,\chi} \circ \eta : \mathcal{O}_{K_n}^\times \rightarrow \mathbb{Z}_p[G_n]$, where $\eta : \mathcal{O}_{K_n}^\times \rightarrow \mathcal{E}_{n,\chi}$ is the natural map and $\varpi : \mathbb{Z}_p(\chi)[\Gamma/\Gamma_n] \rightarrow \mathbb{Z}_p[G_n]$ is defined by $\varpi(\chi(g)v) := [K_0 : k] e_\psi g v$ for all $g \in G$ and $v \in \Gamma/\Gamma_n$. Further, by taking the quotients we obtain a map

$$\Psi_1 : \mathcal{O}_{K_n}^\times / (\mathcal{O}_{K_n}^\times)^M \longrightarrow \mathbb{Z}/M\mathbb{Z}[G_n].$$

Let W_1 be the $\mathbb{Z}_p[G_n]$ -span of $\langle \xi \rangle_M$. We apply Theorem 4.1 to the data

$$m := p^d, \quad W := W_1, \quad \Psi := \Psi_1, \quad \text{and} \quad \mathfrak{c} := \mathfrak{c}_1,$$

We obtain a maximal ideal λ_1 of \mathcal{O}_{K_n} and $u_1 \in (\mathbb{Z}/M\mathbb{Z})^\times$ such that $\text{cl}_p(\lambda_1) = \mathfrak{c}_1^{p^d}$, such that the ideal $\ell_1 := \lambda_1 \cap \mathcal{O}_k$ belongs to \mathcal{L}_{K_n} , and such that for all $w \in W_1$, we have $[w]_{\ell_1} = 0$ and

$$\varphi_{\ell_1}(w) = u_1 p^d \Psi_1(w) \lambda_1. \quad (7.3)$$

We denote by $\bar{\vartheta}_{n,\chi} : \mathcal{E}_{n,\chi} \rightarrow \mathbb{Z}/M\mathbb{Z}[G_n]_\chi$ the morphism obtained from $\vartheta_{n,\chi}$ by taking the quotients. Then from (4.1), we have

$$[\kappa_\varepsilon(\ell_1)]_{\ell_1} = \varphi_{\ell_1}(\xi) = u_1 p^d \Psi_1(\langle \xi \rangle_M) \lambda_1 = u_1 p^d \varpi \circ \bar{\vartheta}_{n,\chi}(\xi') \lambda_1 \quad (7.4)$$

in $\mathcal{I}_{K_n, \ell_1}/M\mathcal{I}_{K_n, \ell_1}$. From (7.4) and (7.2), we deduce that in $(\mathbb{Z}/M\mathbb{Z}[G_n])_\chi$ we have

$$\begin{aligned} \bar{\omega}_{\lambda_1}(\kappa_\varepsilon(\ell_1)) &= u_1 p^d [K_0 : k] \bar{\vartheta}_{n,\chi}(\xi') \\ &= u_1 p^{c_2+d} [K_0 : k] (\gamma_\nu - 1)^{c_1} h'_\chi. \end{aligned} \quad (7.5)$$

Let $i \in \{2, \dots, s+1\}$, and assume that $\lambda_1, \dots, \lambda_{i-1}$ has been constructed. From (d) and (e) we deduce

$$\left(\prod_{j=1}^{i-2} P_j \right) \bar{\omega}_{\lambda_{i-1}}(\kappa_\varepsilon(\mathfrak{a}_{i-1})) = \left(\prod_{j=1}^{i-1} u_j \right) \Delta_i (\gamma_\nu - 1)^{c_1 + \sum_{j=1}^{i-2} c_1^j} h'_\chi \quad (7.6)$$

in $(\mathbb{Z}/M\mathbb{Z}[G_n])_\chi$, with the convention that an empty product is 1 and an empty sum is 0.

Lemma 7.1 *Let L_i be the $\mathbb{Z}_p[G_n]$ -submodule of A_n generated by $\text{cl}_p(\lambda_1), \dots, \text{cl}_p(\lambda_{i-2})$, and let W_i be the $\mathbb{Z}_p[G_n]$ -span of the image of $\kappa_\varepsilon(\mathfrak{a}_{i-1})$ in $K_n^\times / (K_n^\times)^M$. We set $\eta_i := (\gamma_\nu - 1)^{c_1^{i-1}}$, $Z_i := p^{d+c_3}$, and we choose $g_i \in \mathbb{Z}_p[G_n]$ such that the image of g_i and the*

image of P_{i-1} in $\mathbb{Z}_p[G_n]_\chi$ are the same. Then

- (i) $v_{\mathfrak{q}}(\kappa_{\varepsilon}(\mathfrak{a}_{i-1})) \in M\mathbb{Z}$ for all maximal ideal \mathfrak{q} of \mathcal{O}_{K_n} which is prime to \mathfrak{a}_{i-1} .
- (ii) $Z_i \cdot \text{Ann}_{\mathbb{Z}_p[G_n]_\chi}([\text{cl}_p(\lambda_{i-1})]_{L_{i,\chi}}) \subseteq g_i \mathbb{Z}_p[G_n]_\chi$, where $[\text{cl}_p(\lambda_{i-1})]_{L_{i,\chi}}$ is the image of $\text{cl}_p(\lambda_{i-1})$ in $(A_n/L_i)_\chi$.
- (iii) $\mathbb{Z}_p[G_n]_\chi / g_i \mathbb{Z}_p[G_n]_\chi$ is finite.
- (iv) $\# \left(\eta_i \left((\mathcal{I}_{K_n, \ell_{i-1}} / M\mathcal{I}_{K_n, \ell_{i-1}}) / W'_i \right)_\chi \right) \# (A_{n,\chi}) \leq M$, where W'_i is the image of W_i in $\mathcal{I}_{K_n, \ell_{i-1}} / M\mathcal{I}_{K_n, \ell_{i-1}}$ through $w \mapsto [w]_{\ell_{i-1}}$.

Proof. (i) is a direct consequence of (4.1). We have $(A_{\infty, \chi})_{\Gamma_n} \simeq (A_{\infty})_{\Gamma_n, \chi}$, and $(A_{\infty})_{\Gamma_n, \chi}$ is finite by Proposition 5.1. Hence g_i is prime to $1 - \gamma_n$, whence (iii).

Let $\alpha \in \text{Ann}_{\mathbb{Z}_p[G_n]_\chi}([\text{cl}_p(\lambda_{i-1})]_{L_{i,\chi}})$. We can define from τ_n a morphism of $\mathbb{Z}_p[G_n]_\chi$ -modules

$$\tau'_n : (A_n/L_i)_\chi \rightarrow \mathbb{Z}_p[G_n]_\chi / g_i \mathbb{Z}_p[G_n]_\chi,$$

such that the diagram below commutes

$$\begin{array}{ccc} A_{n,\chi} & \xrightarrow{\tau_n} & \bigoplus_{j=1}^s \Lambda_\chi / (P_j, 1 - \gamma_n) \\ \downarrow & & \downarrow \phi \\ (A_n/L_i)_\chi & \xrightarrow{\tau'_n} & \mathbb{Z}_p[G_n]_\chi / g_i \mathbb{Z}_p[G_n]_\chi \end{array},$$

where ϕ is the canonical projection

$$\bigoplus_{j=1}^s \Lambda_\chi / (P_j, 1 - \gamma_n) \longrightarrow \Lambda_\chi / (P_{i-1}, 1 - \gamma_n) \simeq \mathbb{Z}_p[G_n]_\chi / g_i \mathbb{Z}_p[G_n]_\chi.$$

Then $\tau'_n(\mathfrak{c}_{i-1, \chi})^{p^d \alpha} = 0$, i.e. $p^{d+c_3} \alpha \in g_i \mathbb{Z}_p[G_n]_\chi$, so (ii) is verified. From (7.6), and since $c_1 + \sum_{j=1}^{i-2} c_1^j \leq c_1^{i-1}$ (because $2 \leq c_1$), we see that $\eta_i((\mathcal{I}_{K_n, \ell_{i-1}} / M\mathcal{I}_{K_n, \ell_{i-1}}) / W'_i)_\chi$ is cyclic over $\mathbb{Z}/M\mathbb{Z}[G_n]_\chi$, annihilated by $\Delta_i h'_\chi$. The condition (7.1) then implies (iv). \square

Let us apply Lemma 4.1 to the material furnished in Lemma 7.1. There is a morphism of $\mathbb{Z}_p[G_n]$ -modules $\Psi'_i : W_{i,\chi} \rightarrow \mathbb{Z}/M\mathbb{Z}[G_n]_\chi$ such that

$$g_i \Psi'_i \left(\langle \kappa_{\varepsilon}(\mathfrak{a}_{i-1}) \rangle_{M, \chi} \right) \lambda_{i-1, \chi} = Z_i \eta_i [\kappa_{\varepsilon}(\mathfrak{a}_{i-1})]_{\ell_{i-1, \chi}}. \quad (7.7)$$

We define Ψ_i by composing $\varpi \circ \Psi'_i$ with $W_i \rightarrow W_{i,\chi}$. From Lemma 7.1, (i), we can apply Theorem 4.1 to the data

$$F := K_n, \quad m := p^d, \quad \mathfrak{c} := \mathfrak{c}_i, \quad W := W_i, \quad \text{and} \quad \Psi := \Psi_i.$$

There are a maximal ideal λ_i of \mathcal{O}_{K_n} and $u_i \in (\mathbb{Z}/M\mathbb{Z})^\times$ such that $\text{cl}_p(\lambda_i) = \mathfrak{c}_i^{p^d}$ (condition (b)), such that $\ell_i := \lambda_i \cap \mathcal{O}_k$ belongs to \mathcal{L}_{K_n} (condition (a)), and such that for all $w \in W_i$, $[w]_{\ell_i} = 0$ and $\varphi_{\ell_i}(w) = u_i p^d \Psi_i(w) \lambda_i$. By (4.1) we have

$$[\kappa_{\varepsilon}(\mathfrak{a}_i)]_{\ell_{i,\chi}} = \varphi_{\ell_i}(\kappa_{\varepsilon}(\mathfrak{a}_{i-1}))_\chi = u_i p^d \Psi_i(\langle \kappa_{\varepsilon}(\mathfrak{a}_{i-1}) \rangle_M) \lambda_{i,\chi} \quad (7.8)$$

in $(\mathcal{I}_{K_n, \ell_i} / M\mathcal{I}_{K_n, \ell_i})_\chi$. Then in $\mathbb{Z}/M\mathbb{Z}[G_n]_\chi$, by (7.8) and (7.7) we have

$$\begin{aligned} P_{i-1} \bar{\omega}_{\lambda_i}(\kappa_\varepsilon(\mathbf{a}_i)) &= u_i[K_0 : k] p^d g_i \Psi'_i \left(\langle \kappa_\varepsilon(\mathbf{a}_{i-1}) \rangle_{M, \chi} \right) \\ &= u_i[K_0 : k] p^d Z_i \eta_i \bar{\omega}_{\lambda_{i-1}}(\kappa_\varepsilon(\mathbf{a}_{i-1})), \end{aligned}$$

which demonstrates (e). So we can construct recursively the primes $\lambda_1, \dots, \lambda_{s+1}$, and from (d) and (e) we deduce

$$\left(\prod_{j=1}^s P_j \right) \bar{\omega}_{\lambda_{s+1}}(\kappa_\varepsilon(\mathbf{a}_{s+1})) = \left(\prod_{j=1}^{s+1} u_j \right) \Delta_{s+2}(\gamma_\nu - 1)^{c_1 + \sum_{j=1}^s c_1^j} h'_\chi \quad (7.9)$$

in $\mathbb{Z}/M\mathbb{Z}[G_n]_\chi$.

By letting n and M vary, this implies that $\prod_{j=1}^s P_j$ divides $\Delta_{s+2}(\gamma_\nu - 1)^{c_1 + \sum_{j=1}^s c_1^j} h'_\chi$ in Λ_χ . By Proposition 5.1 and since $(A_{\infty, \chi})_{\Gamma_\nu} \simeq (A_{\infty, \Gamma_\nu})_\chi$, $\text{char}(A_{\infty, \chi})$ is prime to $(\gamma_\nu - 1)$. Then we deduce

$$\text{char}(A_{\infty, \chi}) \mid \Delta_{s+2} \text{char}(\mathcal{E}_\infty / \mathcal{C}_\infty)_\chi, \quad (7.10)$$

which proves the assertion (ii) of Theorem 1.1. Now assume $p \notin \{2, 3\}$. Recall that class field theory gives an exact sequence

$$0 \longrightarrow \mathcal{E}_\infty / \mathcal{C}_\infty \longrightarrow \mathcal{U}_\infty / \mathcal{C}_\infty \longrightarrow B_\infty \longrightarrow A_\infty \longrightarrow 0, \quad (7.11)$$

where B_∞ is the Galois group of Ω_∞ / K_∞ , with Ω_∞ the maximal abelian pro- p -extension of K_∞ which is unramified outside of the primes above \mathfrak{p} . Moreover by a result of Gillard ([3, 3.4. Théorème]), the μ -invariant of B_∞ over $\mathbb{Z}_p[[T]]$ vanishes,

$$\mu(B_\infty) = 0. \quad (7.12)$$

Let us denote by $A_{\mathfrak{f}, \infty}$, $\mathcal{E}_{\mathfrak{f}, \infty}$, $\mathcal{C}_{\mathfrak{f}, \infty}$, ..., the various objects attached to $K_{\mathfrak{f}, \infty} := \bigcup_{n=0}^{\infty} k(\mathfrak{f}\mathfrak{p}^n)$. By [2, 2.1 Theorem, p. 109], the divisibility (7.10) and (7.12) applied to $K_{\mathfrak{f}, \infty}$ implies that

$$\text{char}((A_{\mathfrak{f}, \infty})_\xi) = \text{char}(\mathcal{E}_{\mathfrak{f}, \infty} / \mathcal{C}_{\mathfrak{f}, \infty})_\xi \quad \text{and} \quad \mu_\xi(\mathcal{U}_{\mathfrak{f}, \infty} / \mathcal{C}_{\mathfrak{f}, \infty})_\xi = 0, \quad (7.13)$$

for all irreducible \mathbb{C}_p -character ξ of the torsion subgroup $G_{\mathfrak{f}}$ of $\text{Gal}(K_{\mathfrak{f}, \infty} / k)$, and where μ_ξ is the μ -invariant over $\mathbb{Z}_p(\xi)$. Since $(\mathcal{U}_\infty / \mathcal{C}_\infty)_\chi$ is a quotient of $(\mathcal{U}_{\mathfrak{f}, \infty} / \mathcal{C}_{\mathfrak{f}, \infty})_{\tilde{\chi}}$, where $\tilde{\chi}$ is the character of $G_{\mathfrak{f}}$ defined by χ , we deduce from (7.13) that $\mu_\chi(\mathcal{U}_\infty / \mathcal{C}_\infty)_\chi = 0$. By (7.12), the exact sequence (7.11) gives

$$\mu_\chi(\mathcal{E}_\infty / \mathcal{C}_\infty)_\chi = \mu_\chi(\mathcal{U}_\infty / \mathcal{C}_\infty)_\chi = 0 = \mu_\chi(A_{\infty, \chi}). \quad (7.14)$$

By decomposing $H := \text{Gal}(K_{\mathfrak{f}, \infty} / K_\infty)$ into a direct product of cyclic subgroups, we are reduced to the case where H itself is cyclic. Then classical arguments (see [15, section 5]) show that $(A_{\mathfrak{f}, \infty})_H$ is pseudo-isomorphic to A_∞ , and we deduce

$$\lambda((A_{\mathfrak{f}, \infty})_H) = \lambda(A_\infty). \quad (7.15)$$

The cokernel of the norm map $(\mathcal{E}_{\mathfrak{f}, \infty} / \mathcal{C}_{\mathfrak{f}, \infty})_H \rightarrow \mathcal{E}_\infty / \mathcal{C}_\infty$ is annihilated by $\#(H)$, hence $\lambda(\mathcal{E}_\infty / \mathcal{C}_\infty) \leq \lambda(\mathcal{E}_{\mathfrak{f}, \infty} / \mathcal{C}_{\mathfrak{f}, \infty})_H$. Together with (7.13) and (7.15), it implies that

$$\lambda(\mathcal{E}_\infty / \mathcal{C}_\infty) \leq \lambda(\mathcal{E}_{\mathfrak{f}, \infty} / \mathcal{C}_{\mathfrak{f}, \infty})_H = \lambda((A_{\mathfrak{f}, \infty})_H) = \lambda(A_\infty). \quad (7.16)$$

Finally the assertion (i) of Theorem 1.1 follows from (7.16), (7.14) and (7.10).

We draw the attention of the reader to our papers [18] and [17], where we prove that a raw version of Theorem 1.1 (i) holds also for $p \in \{2, 3\}$.

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